Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 851–862
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Reflexive modules on normal surface singularities and representations of the local fundamental group

Trond Stølen Gustavsen[a,*], Runar Ile[b]

[a] *Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N-0316 Oslo, Norway*
[b] *Department of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway*

Received 27 June 2006; received in revised form 27 June 2007; accepted 6 July 2007
Available online 24 August 2007

Communicated by I. Reiten

Abstract

We consider the Riemann–Hilbert correspondence on the complement of a normal surface singularity (X, x). Through a closure operation we obtain a correspondence between the category of finite dimensional representations of the local fundamental group $\pi_1^{\mathrm{loc}}(X, x)$ and the category of left $\mathcal{D}_{X,x}$-modules that are reflexive as $\mathcal{O}_{X,x}$-modules. We show that under this correspondence profinite representations correspond to invariant modules and that these admit a canonical structure as left $\mathcal{D}_{X,x}$-modules. We prove that the fundamental module is an invariant module if and only if (X, x) is a quotient singularity. Finally we investigate some algebraisation aspects.

MSC: 32S40; 13C14; 32S05

1. Introduction

In this article we consider the relationship between reflexive modules on normal surface singularities (X, x) and representations of the local fundamental group $\pi_1^{\mathrm{loc}}(X, x)$.

Our investigation is motivated by two fundamental ideas in the theory of normal surface singularities. The first of these is sometimes called the *Artin program* and was introduced through the seminal articles of Mumford [17] and Artin [3]: To what extent are analytic properties and invariants of normal surface singularities topologically determined, i.e. determined by the link of the singularity? The dual graph Γ of the exceptional fiber in a minimal good resolution, including intersection numbers and genera of the irreducible components, is determined by the link. By a theorem due to W. D. Neumann, the converse also holds. Moreover; if (X, x) is neither a cyclic quotient nor a cusp singularity, then the link is determined by $\pi_1^{\mathrm{loc}}(X, x)$, see [19]. In the case Γ is contractible, an explicit presentation of $\pi_1^{\mathrm{loc}}(X, x)$ in terms of Γ was given by Mumford [17] and Wagreich [21]. Mumford also proved that (X, x) is smooth if and only if $\pi_1^{\mathrm{loc}}(X, x)$ is trivial.

* Corresponding address: Department of Economics, BI Norwegian School of Management, N-0442 Oslo, Norway.
E-mail addresses: stolen@math.uio.no (T.S. Gustavsen), runar.ile@math.uib.no (R. Ile).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.07.002

The second idea is to describe and give characterisations of classes of normal surface singularities in terms of representation theory, which in this context means studying reflexive modules. A normal surface singularity is rational if and only if it has finitely many rank one reflexive modules [17]. A quotient surface singularity has finitely many indecomposable reflexive modules [13]. Auslander and Esnault independently proved the converse: If a normal surface singularity has finitely many reflexive modules, then it is a quotient singularity [4,9].

A combination of these two ideas is considered in this paper: In Theorem 3.1 we prove a Riemann–Hilbert correspondence between reflexive modules with integrable connections and finite dimensional representations of the local fundamental group. Moreover; in Theorem 5.5 we show that under this correspondence profinite representations correspond to invariant modules. These modules arise as the invariants of a $\mathbb{C}$-linear action of the Galois group on a free module on a finite covering of X.

Some natural questions: Which modules admit integrable connections, and how many of them are invariant modules? How many representations give the same reflexive module? We call a normal surface singularity *flat* if all its reflexive modules admit integrable connections. Which singularities are flat? Is (X, x) flat if the fundamental module on (X, x) admits an integrable connection?

Not much seems to be known. Behnke [5] has exhibited a unipotent representation inducing the fundamental module for cusp singularities and conjectured that cusp singularities are flat. All reflexive modules on quotient singularities are invariant modules. In Theorem 5.9 we prove a converse: The fundamental module is an invariant module if and only if the singularity is a quotient singularity.

Kahn [14] has explicitly constructed families of representations for every reflexive module on any simple elliptic singularity. Invoking Kahn's result we show in a second article [12] that all quotients of simple elliptic surface singularities are flat. By the above one is tempted to conjecture that all log-canonical surface singularities are flat.

While Kahn and Behnke only considered connections on the restriction to the smooth locus, by an integrable connection on a reflexive module M on (X, x) we mean an $\mathcal{O}_{X,x}$-linear covariant derivation $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ that is a $\mathbb{C}$-Lie-algebra homomorphism. As a part of the Riemann–Hilbert correspondence we show that the connection ∇ on M can be uniquely extended to a left $\mathcal{D}_{X,x}$-module structure on M. This is surprising since it is known that if X is the cone over the Fermat-cubic (and more generally a cone over any plane curve of genus greater or equal to one), the ring $\mathcal{D}_{X,x}$ of differential operators on X is neither finitely generated nor left (or right) noetherian, see [6,20]. On the other hand the group ring $\mathbb{C}[\pi_1^{\mathrm{loc}}(X, x)]$ *is* noetherian, and this shows in particular that $\mathbb{C}[\pi_1^{\mathrm{loc}}(X, x)]$ and $\mathcal{D}_{X,x}$ are not Morita equivalent even though we have the correspondence between the two subcategories.

At the end we investigate some algebraisation aspects.

2. Notation and preliminaries

In this section we fix notation that will be used throughout the paper. We further define several categories that we will consider.

We work over the field $\mathbb{C}$ of complex numbers, and denote by (X, x) the germ of a normal complex analytic space of dimension two. The local ring of germs of holomorphic functions will be denoted by $\mathcal{O}_{X,x}$. Having chosen a representative X, we denote by U the (smooth) complement in X of the isolated singular point. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on the complex space X and by Θ_X the sheaf of derivations. If M is an $\mathcal{O}_{X,x}$-module, the $\mathbb{C}$-vector space $\mathrm{End}_{\mathbb{C}}(M)$ of $\mathbb{C}$-linear maps is an $\mathcal{O}_{X,x}$-bi-module.

2.1. Connections on modules

Definition 2.1. Let M be a coherent $\mathcal{O}_{X,x}$-module. A *connection* (or covariant derivation) on M is an $\mathcal{O}_{X,x}$-linear map $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ which for all $f \in \mathcal{O}_{X,x}$, $m \in M$ and $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ satisfies the *Leibniz rule*

$$\nabla(D)(fm) = D(f)m + f\nabla(D)(m). \tag{2.1.1}$$

A morphism $\varphi : (M_1, \nabla_1) \to (M_2, \nabla_2)$ is an $\mathcal{O}_{X,x}$-module homomorphism $\varphi : M_1 \to M_2$ such that $\varphi\nabla_1(D) = \nabla_2(D)\varphi$ for all $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ (φ is a *horizontal* map).

A connection $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ is said to be *integrable* if it is a $\mathbb{C}$-Lie-algebra homomorphism.

2.2. Differential operators

Definition 2.2. Let $\mathcal{D}^p_{X,x} := \{\phi \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_{X,x}) | [\phi, a] \in \mathcal{D}^{p-1}_{X,x}$ for all $a \in \mathcal{O}_{X,x}\}$ for $p \geq 0$ and $\mathcal{D}^{-1}_{X,x} = 0$, and set $\mathcal{D}_{X,x} := \cup_p \mathcal{D}^p_{X,x}$.

The sheaf of differential operators $\mathcal{D}_X \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ is defined as follows. For any open $V \subseteq X$ let

$$\mathcal{D}^p_X(V) := \{\phi \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)(V) | [\phi, a] \in \mathcal{D}^{p-1}_X(V) \text{ for all } a \in \mathcal{O}_X(V)\}$$

for $p \geq 0$ and $\mathcal{D}^{-1}_X(V) = 0$. We put $\mathcal{D}_X(V) = \cup_p \mathcal{D}^p_X(V) \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$.

Remark 2.3. Note that $\mathcal{D}^0_{X,x} \cong \mathcal{O}_{X,x}$ and $\mathcal{D}^1_{X,x} \cong \mathcal{O}_{X,x} \oplus \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ as left $\mathcal{O}_{X,x}$-modules. Moreover; $\mathcal{D}_{X,x} = \lim\limits_{\substack{\to \\ x \in V \subseteq X}} \mathcal{D}_X(V)$.

2.3. Categories of reflexive modules

Recall that a coherent $\mathcal{O}_{X,x}$-module M is *reflexive* if the canonical morphism $M \to M^{\vee\vee}$ of the module into its double dual is an isomorphism. When (X, x) is a normal surface singularity, M is reflexive if and only if M is a maximal Cohen–Macaulay $\mathcal{O}_{X,x}$-module, i.e. depth $M = 2$.

We now restrict our attention to modules that are reflexive as $\mathcal{O}_{X,x}$-modules.

Definition 2.4. We define the following categories

$$\mathsf{Ref}_{X,x} = \text{the category of reflexive } \mathcal{O}_{X,x}\text{-modules.}$$

$$\mathsf{Ref}^{\nabla}_{X,x} = \begin{cases} \text{the category of pairs } (M, \nabla) \text{ where } M \text{ is a reflexive} \\ \mathcal{O}_{X,x}\text{-module and } \nabla \text{ is integrable.} \end{cases}$$

$$\mathsf{Ref}^{\mathcal{D}}_{X,x} = \begin{cases} \text{the category of left } \mathcal{D}_{X,x}\text{-modules that are} \\ \text{reflexive as } \mathcal{O}_{X,x}\text{-modules.} \end{cases}$$

There is a natural functor $\mathsf{Ref}^{\mathcal{D}}_{X,x} \longrightarrow \mathsf{Ref}^{\nabla}_{X,x}$. The left $\mathcal{D}_{X,x}$-module structure on M is given by a map $\psi : \mathcal{D}_{X,x} \to \mathrm{End}_{\mathbb{C}}(M)$, and for every $P \in \mathcal{D}_{X,x}$ we get $\psi_P \in \mathrm{End}_{\mathbb{C}}(M)$. We claim that restriction of ψ to $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \subset \mathcal{D}_{X,x}$ defines a connection ∇. For any $P_1, P_2 \in \mathcal{D}_{X,x}$, we have $\psi_{P_1 P_2} = \psi_{P_1}\psi_{P_2}$. Hence

$$\psi_{P_1}\psi_{P_2} = \psi_{[P_1,P_2]+P_2P_1} = \psi_{[P_1,P_2]} + \psi_{P_2P_1} = \psi_{[P_1,P_2]} + \psi_{P_2}\psi_{P_1}.$$

In particular, with $P_1 = D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ and $P_2 = f \in \mathcal{O}_{X,x}$, we find that ∇ satisfies the Leibniz rule, and with $P_i = D_i$, $i = 1, 2$, we get that ∇ is integrable.

2.4. Representatives

A coherent $\mathcal{O}_{X,x}$-module M may be represented by a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules, for a representative X of (X, x). The following property is important in this paper:

Lemma 2.5 ([10, II.5.29 on p. 141]). *Let $\mathcal{M}$ be a sheaf of reflexive $\mathcal{O}_X$-modules (i.e. $\mathcal{M} \cong \mathcal{M}^{\vee\vee}$). Then the restriction map $\mathcal{M}(X) \to \mathcal{M}(X \setminus \{x\})$ is an isomorphism. Moreover $\mathcal{M}_{|X\setminus\{x\}}$ is locally free if $X \setminus \{x\}$ is smooth of dimension two.*

We may also consider representatives for connections:

Lemma 2.6. *Assume that M is a coherent $\mathcal{O}_{X,x}$-module, and let $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ be an integrable connection. Then there exists an $\mathcal{O}_X$-linear map $\nabla_X : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ such that ∇ equals the composition of $(\nabla_X)_x : \Theta_{X,x} \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})_x$ with the natural map $\mathcal{E}nd_{\mathbb{C}}(\mathcal{M})_x \to \mathrm{End}_{\mathbb{C}}(M)$.*

Proof. Since $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x})$ is coherent, we may assume that the sheaf of derivations Θ_X on X is generated by global sections $D_1, \ldots, D_s$ in $\Theta_X(X)$. If $m_1, \ldots, m_t \in \mathcal{M}(X)$ are generating sections for $\mathcal{M}$, we may, by restricting

X if necessary, assume that $\nabla_{ij} := \nabla(D_i)(m_j)$ are represented in $\mathcal{M}(X)$. Then applying the Leibniz rule, $\nabla(D_i)(fm_j) = D(f)m_j + f\nabla(D_i)(m_j)$, we construct an $\mathcal{O}_X$-linear map $\nabla_X : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ by defining $\nabla_X(D_i)(m_j) := \nabla_{ij}$. $\square$

Note that $\nabla_X(V)$ will satisfy the Leibniz rule and be a Lie-algebra homomorphism for all open $V \subseteq X$, hence we will call ∇_X an integrable connection on $\mathcal{M}$. Restricting to the smooth U, one easily sees that $\nabla : \Theta_U \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}_{|U})$ corresponds to a connection $\nabla'_U : \mathcal{M}_{|U} \to \Omega^1_U \otimes \mathcal{M}_{|U}$ in the usual sense, since $\mathcal{M}_{|U}$ is locally free. The correspondence is given as follows. If D is a local section of Θ_U, we get a section

$$\mathcal{M}_{|U} \xrightarrow{\nabla'} \Omega^1_U \otimes \mathcal{M}_{|U} \xrightarrow{D\otimes 1} \mathcal{O}_U \otimes \mathcal{M}_{|U} \cong \mathcal{M}_{|U}$$

of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{M}_{|U})$, and this defines the $\mathcal{O}_U$-linear map $\nabla_U : \Theta_U \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}_{|U})$. Note, however, that $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ does not in general correspond to a connection $\nabla : M \to \Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} M$.

2.5. *The local fundamental group*

We may always choose a representative X of a normal surface singularity (X, x), such that $X \setminus \{x\}$ is connected and $(X, x) \subset (\mathbb{C}^n, 0)$. If $\varepsilon > 0$ is small and B_ε is a ball in $\mathbb{C}^n$ of radius ε, then $L := X \cap \partial B_\varepsilon$ is a smooth, compact, connected and oriented real 3-manifold, called the link of (X, x), see [17]. The isomorphism class of the link L is independent of (small) ε.

Definition 2.7. We define the local fundamental group $\pi_1^{\mathrm{loc}}(X, x) := \pi_1(L)$.

After substituting $X \cap B_\varepsilon$ for X for small enough ε, X is homeomorphic to the cone over L, e.g. $\pi_1^{\mathrm{loc}}(X, x) = \pi_1(U)$. We will always assume that X is such a *good* representative.

Definition 2.8. We define $\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)}$ to be the category of finite dimensional representations of $\pi_1^{\mathrm{loc}}(X, x)$ over the complex numbers.

3. The Riemann–Hilbert correspondence

In this section we define functors

$$F : \mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \to \mathsf{Ref}^{\nabla}_{X,x} \quad \text{and} \quad G : \mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \to \mathsf{Ref}^{\mathcal{D}}_{X,x}$$

and we prove the following theorem:

Theorem 3.1. *The functors F and G give equivalences*

$$\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \simeq \mathsf{Ref}^{\nabla}_{X,x} \simeq \mathsf{Ref}^{\mathcal{D}}_{X,x}$$

between the category of finite dimensional representations of the local fundamental group of (X, x) with the category of pairs (M, ∇) of a reflexive $\mathcal{O}_{X,x}$-module M with integrable connection ∇ (and horizontal maps), and with the category of left $\mathcal{D}_{X,x}$-modules M that are also reflexive as $\mathcal{O}_{X,x}$-modules.

Example 3.2. It is not possible to extend the equivalence in Theorem 3.1 to an equivalence between the category of $\mathbb{C}[\pi_1^{\mathrm{loc}}(X, x)]$-modules and the category of left $\mathcal{D}_{X,x}$-modules.

Let X denote the cone over the Fermat cubic, that is

$$X = V(x^3 + y^3 + z^3) \subset \mathbb{C}^3.$$

The fundamental group $\pi_1^{\mathrm{loc}}(X, x)$ is generated by three elements α, β and γ with the relations

$$[\alpha, \gamma] = 1, \qquad [\beta, \gamma] = 1 \quad \text{and} \quad [\alpha, \beta] = \gamma^3,$$

see [14, 6.2]. We claim that the group ring $\mathbb{C}[\pi_1^{\mathrm{loc}}(X, x)]$ is left noetherian.[1] We consider $R = \mathbb{C}[\pi_1^{\mathrm{loc}}(X, x)]$ as a graded ring by giving α and β degree one, and γ degree zero. Then R_0 is the commutative ring

[1] We thank Eivind Eriksen for providing us with this argument.

$\mathbb{C}[\alpha^{-1}\beta, \beta^{-1}\alpha, \gamma, \gamma^{-1}]$. We have $R_i = \alpha^i R_0 = R_0\alpha^i$. Then from Proposition 3.4 of [18], it follows that R is left (and right) noetherian. On the other hand $\mathcal{D}_{X,x}$ is not left noetherian, see [6].

The ring $\mathcal{D}_{X,x}$ of differential operators is in general not generated in degree one for the cone (X, x) over any plane curve of genus ≥ 1, see [20]. Still a left $\mathcal{D}_{X,x}$-module structure on a reflexive $\mathcal{O}_{X,x}$-module is determined by the action of the degree one operators (i.e. the derivations) by Theorem 3.1.

Remark 3.3. We note that $\mathsf{Ref}^{\nabla}_{X,x}$ is an abelian category (by Theorem 3.1) in contrast to $\mathsf{Ref}_{X,x}$.

3.1. Definition of the functors F and G

For the purpose of clarifying the exposition, we define two intermediate categories Vect^{∇}_U and $\mathsf{Vect}^{\mathcal{D}}_U$ where U again denotes the complement of the singular point in a fixed small representative X and $i : U \hookrightarrow X$ denotes the inclusion.

The objects in Vect^{∇}_U are pairs $(\mathcal{E}, \nabla')$ where $\mathcal{E}$ is a locally free sheaf on U and ∇' is an integrable connection $\nabla' : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ (in the usual sense.) The morphisms in the category are the horizontal maps. The objects of $\mathsf{Vect}^{\mathcal{D}}_U$ are left $\mathcal{D}_U$-modules $\mathcal{E}$ such that $\mathcal{E}$ is a locally free $\mathcal{O}_U$-module and the morphisms are $\mathcal{D}_U$-module homomorphisms.

Lemma 3.4. *There are equivalences of categories*

$$\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \simeq \mathsf{Vect}^{\nabla}_U \simeq \mathsf{Vect}^{\mathcal{D}}_U.$$

In particular; if $X' \subset X$ is a smaller good *representative of the singularity, then the corresponding inclusion $U' \subset U$ induces an equivalence of categories $\mathsf{Vect}^{\nabla}_{U'} \cong \mathsf{Vect}^{\nabla}_U$ and $\mathsf{Vect}^{\mathcal{D}}_{U'} \simeq \mathsf{Vect}^{\mathcal{D}}_U$.*

Proof. The statement in the lemma is a consequence of the classical Riemann–Hilbert correspondence on U. For the last equivalence Theorem 1.2.12 in [7] applies since U is non-singular and $\mathcal{E}$ is locally free. □

Because of the equivalence $\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \xrightarrow{\sim} \mathsf{Vect}^{\nabla}_U$, we will define F as a functor $F : \mathsf{Vect}^{\nabla}_U \to \mathsf{Ref}^{\nabla}_{X,x}$.

Lemma 3.5. *If $(\mathcal{E}, \nabla')$ is in Vect^{∇}_U then $\mathcal{M} := i_*\mathcal{E}$ is a sheaf of reflexive modules on X.*

Proof. From Lemma 3.4, $\mathcal{E} \cong \mathbb{V} \otimes \mathcal{O}_U$ where $\mathbb{V} = \ker \nabla'$. We claim that $i_*(\mathbb{V} \otimes \mathcal{O}_U)$ is coherent for any local system $\mathbb{V}$ on U.

We consider $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ on $\widetilde{X} \setminus E \cong U$ where $\pi : \widetilde{X} \to X$ is the minimal good resolution of X and $E = \pi^{-1}(x)$ is the exceptional set. The sheaf $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ may be extended to a locally free sheaf $\mathcal{E}_1$ on $\widetilde{X}$. (See for instance the "Key lemma" on page 547 of [15].) By Grauert's Direct Image Theorem $\pi_*\mathcal{E}$ and hence $(\pi_*\mathcal{E})^{\vee\vee}$ is coherent. By Lemma 2.5, we have that $i_*(\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U) \cong i_*((\pi_*\mathcal{E}_1)^{\vee\vee}_{|U}) = (\pi_*\mathcal{E}_1)^{\vee\vee}$ is coherent. In particular, $\mathcal{M} = i_*\mathcal{E}$ is coherent.

By Lemma 2.5 applied to $\mathcal{M}$ and $\mathcal{O}_X$, we have that

$$\begin{aligned} \mathcal{M}^{\vee\vee} &\cong \mathcal{H}om(\mathcal{H}om(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X) \cong \mathcal{H}om(\mathcal{H}om(i_*\mathcal{E}, i_*\mathcal{O}_U), i_*\mathcal{O}_U) \\ &\cong i_*\mathcal{H}om(\mathcal{H}om(\mathcal{E}, \mathcal{O}_U), \mathcal{O}_U) \cong i_*\mathcal{E} \cong \mathcal{M} \end{aligned}$$

so $\mathcal{M}$ is reflexive. □

Assume we have a connection $\nabla' : \mathcal{E} \to \Omega^1_U \otimes \mathcal{E}$. As in Section 2.4 there is a corresponding $\nabla_U : \Theta_U \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$. We define $F(\mathcal{E}, \nabla') = (M, \nabla)$ where $M = (i_*\mathcal{E})_x$ and $\nabla = (i_*\nabla_U)_{(x)}$ is the composition of $(i_*\nabla_U)_x : \Theta_{X,x} = (i_*\Theta_U)_x \to (i_*\mathcal{E}nd_{\mathbb{C}}(\mathcal{E}))_x = \mathcal{E}nd_{\mathbb{C}}(i_*\mathcal{E})_x$ and the natural map $\mathcal{E}nd_{\mathbb{C}}(i_*\mathcal{E})_x \to \mathrm{End}_{\mathbb{C}}(M)$. Since ∇_U, and hence $(i_*\nabla_U)_x$, satisfies the Leibniz rule, it follows that $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ is a connection. Similarly, since ∇' is integrable, ∇_U is a $\mathbb{C}$-Lie-algebra homomorphism, and it follows that $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$ is a $\mathbb{C}$-Lie-algebra homomorphism.

Since i_* and passing to the stalk are functors, it is clear that the construction is functorial. Hence F is well defined.

Lemma 3.6. *The sheaf $\mathcal{D}_X = \cup\mathcal{D}^p_X$ of differential operators satisfies the following properties:*

(1) *$\mathcal{D}^p_X$ is a sheaf of reflexive $\mathcal{O}_X$-modules and $\mathcal{D}^p_{X,x}$ is a reflexive $\mathcal{O}_{X,x}$-module*
(2) $i_*\mathcal{D}_U = \mathcal{D}_X$
(3) *i_* defines a functor from $\mathsf{Vect}^{\mathcal{D}}_U$ to the category $\mathsf{Ref}^{\mathcal{D}}_X$ of sheaves of left $\mathcal{D}_X$-modules that are $\mathcal{O}_X$-reflexive.*

Proof. We have that $\mathcal{D}_X^p = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_X^p, \mathcal{O}_X)$ where $\mathcal{P}_X^p$ is a coherent sheaf, see [11, EGA IV$_4$ 16.7.3 and 16.8.4]. It follows that $\mathcal{D}_X^p$ is reflexive and coherent. From this it follows by Lemma 2.5 that $i_*\mathcal{D}_U^p = \mathcal{D}_X^p$ and hence $i_*\mathcal{D}_U = \mathcal{D}_X$. The third assertion follows from Lemma 3.5, since trivially, we have that $i_*\mathcal{E}$ is a sheaf of left $i_*\mathcal{D}_U$-modules. $\square$

Composing i_* with the stalk functor, we obtain by Lemmas 3.6 and 3.5 a functor $G : \mathsf{Vect}_U^{\mathcal{D}} \to \mathsf{Ref}_{X,x}^{\mathcal{D}}$. Note that the action of $\mathcal{D}_{X,x}$ on $G(\mathcal{E})$ is given through the natural map $\mathcal{E}nd_{\mathbb{C}}(i_*\mathcal{E})_x \to \mathrm{End}_{\mathbb{C}}(M)$.

3.2. Proof of Theorem 3.1

For the proof of Theorem 3.1 we will use the following lemma. Denote by $\mathcal{A}_X^p$ the subsheaf of algebras $\mathcal{A}_X^p \subset \mathcal{D}_X$ generated by $\mathcal{D}_X^p$.

Lemma 3.7. *Given a $\mathcal{D}_{X,x}$-module M (where $M = \mathcal{M}_x$), there exists a decreasing sequence of open subsets $V_p \ni x$ such that M as an $\mathcal{A}_{X,x}^p$-module is represented by a sheaf of $\mathcal{A}_{X|V_p}^p$-modules $\mathcal{M}_{|V_p}$ on V_p.*

Proof. Using that $\mathcal{D}_{X,x}^p$ is a coherent $\mathcal{O}_{X,x}$-module, see Lemma 3.6, we may consider generators $P_1, \ldots, P_n$ for $\mathcal{D}_{X,x}^p$ and generators $m_1, \ldots, m_s$ of M represented on some V. Let $m \in M$. Then $m = \sum f_j m_j$. Consider $P_i m = \sum P_i(f_j m_j) = \sum (f_j P_i + [P_i, f_j]) m_j$. Since $[P_i, f_j] \in \mathcal{D}_{X,x}^{p-1}$ we see by induction that the endomorphism $P_i\cdot$ is determined by its values on $m_1, \ldots, m_s$. From this we conclude that M as an $\mathcal{A}_{X,x}^p$-module is represented by a sheaf of $\mathcal{A}_{X|V_p}^p$-modules $\mathcal{M}_{|V_p}$ on some open $V_p \subset V \cap V_{p-1}$. $\square$

Proof of Theorem 3.1. To show that F is an equivalence of categories, we first prove that for any object (M, ∇) in $\mathsf{Ref}_{X,x}^{\nabla}$ there is an object $(\mathcal{E}, \nabla')$ in Vect_U^{∇} such that $F(\mathcal{E}, \nabla') \cong (M, \nabla)$. If (M, ∇) is in $\mathsf{Ref}_{X,x}^{\nabla}$, then from Lemma 2.6 we have a sheaf $\mathcal{M}$ and a connection $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$. Restricting to U and letting $\mathcal{E} = \mathcal{M}_{|U}$ we get $\nabla_U : \Theta_U \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{E})$, and as in Section 2.4, this corresponds to $\nabla' : \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$. It follows that $F(\mathcal{E}, \nabla') \cong ((i_*\mathcal{E})_x, (i_*\nabla_U)_{(x)}) \cong (M, \nabla)$.

If $\varphi \in \mathrm{Hom}((\mathcal{E}_1, \nabla_1'), (\mathcal{E}_2, \nabla_2'))$ and $F(\varphi) = 0$, then since $i_*\mathcal{E}_1$ and $i_*\mathcal{E}_2$ are coherent, there is an open $V \subseteq X$ such that $i_*\varphi_{|V} = 0$. We get that $\varphi_{|V\cap U} = 0$, hence $\varphi = 0$ and $F : \mathrm{Hom}((\mathcal{E}_1, \nabla_1'), (\mathcal{E}_2, \nabla_2')) \to \mathrm{Hom}((M_1, \nabla_1), (M_2, \nabla_2))$ is injective. To prove surjectivity, note from Lemma 2.6 that $\psi \in \mathrm{Hom}((M_1, \nabla_1), (M_2, \nabla_2))$ may be represented by a $\psi' : (\mathcal{M}_1, \nabla_1) \to (\mathcal{M}_2, \nabla_2)$ on some open $V \subseteq X$. Restricting, we define $\mathcal{E}_i^V = \mathcal{M}_{i|V\cap U}$ and obtain a horizontal $\varphi^V : \mathcal{E}_1^V \to \mathcal{E}_2^V$. By Lemma 3.4, we get a unique $\varphi : (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ extending φ^V. One checks that $F(\varphi) = \psi$.

To prove that G is an equivalence of categories, we first remark that by Lemma 3.7 there is an $\mathcal{A}_{X|V_1}^1$-module $\mathcal{M}_{|V_1}$ on $V = V_1$ such that $\mathcal{M}$ represents a given M in $\mathsf{Ref}_{X,x}^{\mathcal{D}}$ as an $\mathcal{O}_X$-module. But since $V \cap U$ is smooth, $\mathcal{A}_{V\cap U}^1 = \mathcal{A}_{X|V\cap U}^1 = \mathcal{D}_{V\cap U}$, see Section I.2 in [7]. Thus $\mathcal{E}^V = \mathcal{M}_{|V\cap U}$ is a left $\mathcal{D}_{V\cap U}$-module. By Lemma 3.4 we may extend to a $\mathcal{D}_U$-module $\mathcal{E}$. We need to show that $G(\mathcal{E})$ and M are isomorphic as $\mathcal{D}_{X,x}$-modules. In other words we must show that the action of $\mathcal{D}_{X,x}$ on M is determined by the action of $\mathcal{D}_{V\cap U}$ on $\mathcal{E}^V = \mathcal{M}_{|V\cap U}$. Let $P \in \mathcal{D}_{X,x}$. Then $P \in \mathcal{A}_{X,x}^p$ for some p, and we get from Lemma 3.7 that P is represented by P_V on some $V := V_p \subseteq V_1$. Let $m \in M$. Then m is represented by an $m_V \in \mathcal{M}(V)$. Restricting V if necessary we have $Pm = (P_V m_V)_x$. But $P_V m_V = P_{V|U} m_{V|U}$ since $\mathcal{M}(V \cap U) = \mathcal{M}(V)$, by Lemma 2.5. Hence the action of P on M is uniquely determined by the action of $\mathcal{D}_{V\cap U}$ on $\mathcal{E}^V = \mathcal{M}_{|V\cap U}$.

Injectivity of $G : \mathrm{Hom}_{\mathcal{D}_U}(\mathcal{E}_1, \mathcal{E}_2) \to \mathrm{Hom}_{\mathcal{D}_{X,x}}(M_1, M_2)$ follows in the same way as for F. To prove surjectivity, we may as above extend a $\psi \in \mathrm{Hom}_{\mathcal{D}_{X,x}}(M_1, M_2)$ to an $\mathcal{A}_{X|V}^1$-linear map $\psi_V : (i_*\mathcal{E}_1)_{|V} \to (i_*\mathcal{E}_2)_{|V}$. Restricting ψ_V to $V \cap U$ gives an $\mathcal{A}_{X|V\cap U}^1 = \mathcal{D}_{V\cap U}$-linear map. By Lemma 3.4 there is an extension $\psi_U : \mathcal{E}_1 \to \mathcal{E}_2$ and $(i_*\psi_U)_x = \psi$. $\square$

3.3. Homomorphisms and tensor products

Let $R = \mathcal{O}_{X,x}$. If (M, ∇) and (M', ∇') are two R-modules with connections, the R-module $\mathrm{Hom}_R(M, M')$ has a natural integrable connection ∇'' given by $\nabla''(D) = [\varphi \mapsto \varphi \circ \nabla(D) - \nabla'(D) \circ \varphi]$. This connection has the property that the horizontal sections are precisely the horizontal maps.

Recall that if M and M' are reflexive R-modules, then $M \otimes_R M'$ need not be reflexive. We thus define $M[\otimes]M'$ to be $(M \otimes_R M')^{\vee\vee}$. This is the correct tensor product in the category $\mathsf{Ref}_{X,x}$ and has the right universal properties.

There is likewise a tensor product in the category $\mathsf{Ref}^{\nabla}_{X,x}$ denoted $(M'', \nabla'') = (M, \nabla)[\otimes](M', \nabla')$ where $M'' = M[\otimes]M'$ and where ∇'' is defined as follows: There is a natural connection on ∇_1 on $M \otimes_R M'$ such that $\nabla_1(D)(m \otimes m') = \nabla(D)(m) \otimes \nabla''(D)(m)$ for all $D \in \mathrm{Der}_{\mathbb{C}}(R)$. We define ∇'' as the natural connection on $(M \otimes_R M')^{\vee\vee}$ induced by ∇_1.

4. Reflexive modules and Galois coverings

In this section we will consider Galois coverings $\pi : (Y, y) \to (X, x)$ and compare the categories $\mathsf{Ref}_{Y,y}$ and $\mathsf{Ref}_{X,x}$ of reflexive modules on (Y, y) and (X, x).

4.1. Galois coverings

We define Galois coverings and give some basic properties.

Definition 4.1. A surjective map of germs $\pi : (Y, y) \to (X, x)$ of normal surface singularities will be said to be a Galois covering of (X, x) if it is finite, étale on the complement $Y \setminus \{y\} \to X \setminus \{x\}$ (for some choice of representatives) and the number of elements in $G(Y/X) := \mathrm{Aut}((Y, y)/(X, x))$ equals the number of elements in the fiber over a point $z \in X \setminus \{x\}$.

Galois coverings $\pi : (Y, y) \to (X, x)$ are in natural correspondence with Galois coverings of the smooth manifold $X \setminus \{x\}$ (restricting if necessary.) Noting SGA1 XII.5.4 (see [1]), we have the following properties:

Proposition 4.2. *Let $\pi : (Y, y) \to (X, x)$ be a finite map. Let $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$, and let K and L be the quotient fields of R and S respectively.*

(1) *The map π is a Galois covering if and only if L is a Galois extension of K, S is the integral closure of R in L and every height one prime ideal $\mathfrak{p}$ in R is unramified in S.*
(2) *If π is a Galois covering, G acts freely on $Y \setminus \{y\}$ (after restricting the representative, if necessary) and $R = S^G$.*
(3) *If π is a Galois covering there is a natural exact sequence of groups*
$$1 \to \pi_1^{\mathrm{loc}}(Y, y) \xrightarrow{\pi_*} \pi_1^{\mathrm{loc}}(X, x) \to G(Y/X) \to 1$$
and Galois coverings of (X, x) are in natural one-to-one correspondence with normal subgroups of $\pi_1^{\mathrm{loc}}(X, x)$ of finite index. Hence, if $(Z, z) \to (Y, y)$ is a Galois covering, there is an exact sequence
$$1 \to G(Z/Y) \to G(Z/X) \to G(Y/X) \to 1.$$

4.2. Reflexive modules with Galois group actions

We fix a Galois covering $\pi : (Y, y) \to (X, x)$ and consider $S = \mathcal{O}_{Y,y}$-modules M that have an action of $G = G(Y/X)$ *compatible* with the action of G on $\mathcal{O}_{Y,y}$, i.e. an action of G on M satisfying $\sigma(sm) = \sigma(s)\sigma(m)$ for $\sigma \in G$ where $\sigma(s)$ refers to the given action on S.

The skew group ring $S[G]$ is the free S-module with the elements of G as basis and with multiplication given by $(s_1\sigma_1)(s_2\sigma_2) = s_1\sigma_1(s_2)\sigma_1\sigma_2$, where the s_i are in S and the σ_i are in G. It will be convenient to consider modules with compatible G-action as modules over the skew group ring.

We will denote the category of $S[G]$-modules that are reflexive as S-modules by $\mathsf{Ref}^{G}_{Y,y}$.

Proposition 4.3. *Let $\pi : (Y, y) \to (X, x)$ be a Galois covering. Let $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$.*

(1) *Let M be an indecomposable reflexive R-module, and assume that $(\pi^*M)^{\vee\vee}$ is S-free. Then M is a direct summand of S considered as an R-module.*
(2) *Let N be a reflexive S-module. Then every R-summand of N is a reflexive R-module.*
(3) *Let N be an $S[G]$-module that is reflexive as S-module. Then the module $N^G = \pi_*N^G$ is a direct summand of π_*N, and hence a reflexive $R = S^G$-module.*

Proof. The proof is left to the reader. □

It follows from Proposition 4.3 that we have a functor

$$I : \mathsf{Ref}^G_{Y,y} \to \mathsf{Ref}_{X,x}$$

given by $I(N) = \pi_* N^G$. We will show that it is an additive equivalence of categories.

Proposition 4.4. *Let $\pi : (Y, y) \to (X, x)$ be a Galois covering, with $G = G(Y/X)$. Let $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$. The functor I is an additive equivalence of categories, and an inverse equivalence is given by $I^{(-1)}(M) = (S \otimes_R M)^{\vee\vee}$.*

Proof. We have a functor $I^{(-1)}$ from the category of R-modules to the category of $S[G]$-modules given by

$$I^{(-1)}(M) = (S \otimes_R M)^{\vee\vee} = \mathrm{Hom}_S(\mathrm{Hom}_S(S \otimes_R M, S), S).$$

The action of G on $(S \otimes_R M)^{\vee\vee}$ is induced by the action on $S \otimes_R M$ given by $\sigma(\varphi)(\psi) = \sigma(s) \otimes m$ for $\sigma \in G$ and the natural action on the homomorphisms. We claim that I and $I^{(-1)}$ set up an equivalence of categories.

For an $S[G]$-module N that is reflexive as S-module, we first prove that $I^{(-1)} \circ I(N) = (S \otimes_R N^G)^{\vee\vee} \cong N$ by a natural isomorphism. We have a product map $S \otimes_R N \to N$, given by $s \otimes n \mapsto sn$. Composing with the inclusion $N^G \subset N$ tensorised with S over R, we get a map $S \otimes_R N^G \to N$. Dualising twice we get $\varphi : (S \otimes_R N^G)^{\vee\vee} \to N$. Since we have (partially) internal Hom groups in $\mathsf{Ref}^G_{Y,y}$, φ is an isomorphism if and only if the restriction to $Y \setminus \{y\}$, $\varphi : \left(\pi^* N^G\right)_{|Y\setminus\{y\}} \to N_{|Y\setminus\{y\}}$ is an isomorphism. But since G acts freely on $Y \setminus \{y\}$ (see Proposition 4.2), this follows from a well known equivalence between the category of vector bundles with G-action and the category of vector bundles on the quotient. Hence $I^{(-1)}$ is full.

If $\varphi : M_1 \to M_2$ is in $\mathsf{Ref}_{X,x}$ such that $(\pi^*\varphi)^{\vee\vee} = 0$, then $\pi^*\varphi_{|Y\setminus\{y\}} = 0$ so $\varphi_{|U} = 0$ and since we have internal Hom groups in $\mathsf{Ref}_{X,x}$, $\varphi_{|U} = 0$ if and only if $\varphi = 0$, i.e. $I^{(-1)}$ is faithful. It follows that for any reflexive R-module M, $I \circ I^{(-1)}(M) = ((S \otimes_R M)^{\vee\vee})^G$ is naturally isomorphic to M. $\square$

5. Profinite representations

In this section, we consider profinite representations of the local fundamental group and reflexive modules that are free on a Galois covering.

Definition 5.1. A representation $\rho : \pi_1^{\mathrm{loc}}(X, x) \to \mathrm{Gl}(n, \mathbb{C})$ is *profinite* if the image is a finite group. We define $\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)}$ to be the full subcategory of $\mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)}$ of profinite representations.

Definition 5.2. Let A be the composition of the functors

$$\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)} \to \mathsf{Rep}_{\pi_1^{\mathrm{loc}}(X,x)} \xrightarrow{F} \mathsf{Ref}^{\nabla}_{X,x} \to \mathsf{Ref}_{X,x},$$

and define $\mathsf{Ref}^{\mathsf{i}}_{X,x}$ to be the full subcategory of $\mathsf{Ref}_{X,x}$ of reflexive modules M such that $(\pi^*M)^{\vee\vee}$ is free for some Galois covering $\pi : (Y, y) \to (X, x)$. We define $\mathsf{Ref}^{\mathsf{i},\nabla}_{X,x}$ to be the full subcategory of $\mathsf{Ref}^{\nabla}_{X,x}$ consisting of pairs (M, ∇) such $(M, \nabla) \cong F(\rho)$ for some ρ in $\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)}$.

Let $\pi : (Y, y) \to (X, x)$ be a Galois covering. We note that $G(Y/X)$ acts on $(\pi^*M)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$. Composing with $\pi_1^{\mathrm{loc}}(X, x) \to G(Y/X)$ we get that $(\pi^*M)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$ forms a directed system of representations of the fundamental group, when π runs through all (finite) Galois coverings.

Definition 5.3. We define $B : \mathsf{Ref}^{\mathsf{i}}_{X,x} \to \mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)}$ by

$$B(M) = \lim_{\substack{\to \\ \pi}} (\pi^*M)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}.$$

It is important to restrict the definition of B to $\mathsf{Ref}^{\mathsf{i}}_{X,x}$, since $\lim_{\substack{\to \\ \pi}} (\pi^*M)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$ in general is not finite dimensional, see Remark 5.10.

If ρ is a representation of the local fundamental group, we denote by V_ρ the corresponding $\pi_1^{\mathrm{loc}}(X,x)$-module and by $\mathbb{V}_\rho$ the corresponding local system on U. Suppose that ρ is profinite. We denote by $\pi_\rho : (Y,y) \to (X,x)$ the Galois covering corresponding to the quotient $\pi_1^{\mathrm{loc}}(X,x) \twoheadrightarrow \operatorname{im}\rho$ of the local fundamental group (see Proposition 4.2).

Lemma 5.4. *We have a natural isomorphism*

$$A(V_\rho) \cong \left((\pi_\rho)_*\left(V_\rho \otimes \mathcal{O}_{Y,y}\right)\right)^{\operatorname{im}\rho}.$$

Proof. This description of A follows by the definition of F since $(\pi_*(V_\rho \otimes \mathcal{O}_Y))_{|U}^{\operatorname{im}\rho} = \mathbb{V}_\rho \otimes \mathcal{O}_U$ and $(\pi_*(V_\rho \otimes \mathcal{O}_{Y,y}))^{\operatorname{im}\rho}$ is a reflexive module. $\square$

Theorem 5.5. *There is a natural isomorphism of the composed functor BA with the identity functor on $\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)}$, and for every object M in $\mathsf{Ref}^{\mathsf{i}}_{X,x}$ we have $AB(M) \cong M$ (not natural). There is an equivalence of categories*

$$\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)} \simeq \mathsf{Ref}^{\mathsf{i},\nabla}_{X,x}.$$

Proof. Suppose $\varphi : V_{\rho_1} \to V_{\rho_2}$ in $\mathsf{Rep}_{\widehat{\pi}_1^{\mathrm{loc}}(X,x)}$. Then $A(\varphi)$ is given as follows: Let $\pi_\varphi : (Y,y) \to (X,x)$ be a Galois covering dominating π_{ρ_1} and π_{ρ_2}. Then $\operatorname{im}\rho_1$ and $\operatorname{im}\rho_2$ are quotients of $G = G(Y_\varphi/X)$ and G acts on V_{ρ_i}. We get $A(\varphi) = ((\pi_\varphi)_*(V_{\rho_1} \otimes \mathcal{O}_{Y,y} \xrightarrow{\varphi\otimes\mathrm{id}} V_{\rho_2} \otimes \mathcal{O}_{Y,y}))^G$. From Proposition 4.4 we get that $(\pi_\varphi^*((\pi_\varphi)_*(\varphi \otimes_{\mathbb{C}} \mathrm{id}_{\mathcal{O}_{Y,y}})^G)^{\vee\vee}) = \varphi \otimes_{\mathbb{C}} \mathrm{id}_{\mathcal{O}_{Y,y}}$. Since for any Galois covering $\pi : (Z,z) \to (Y,y)$, we have that $(\pi^*(V_{\rho_i} \otimes \mathcal{O}_{Y,y}))^{\vee\vee}$ is naturally isomorphic to $V_{\rho_i} \otimes \mathcal{O}_{Z,z}$, we obtain isomorphisms $B(A(V_{\rho_i})) \cong V_{\rho_i}$ compatible with $B(A(\varphi))$.

Assume that M is such that $(\pi^*M)^{\vee\vee}$ is free. Then $B(M) \cong (\pi^*M)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y} =: V_\rho$. The claim is that $M \cong A(V_\rho)$. From Proposition 4.4 $(\pi^*M)^{\vee\vee}$ is an $\mathcal{O}_{Y,y}[G]$-module, and since it is free as $\mathcal{O}_{Y,y}$-module it follows that it is projective as an $\mathcal{O}_{Y,y}[G]$-module. Thus the surjection $(\pi^*M)^{\vee\vee} \twoheadrightarrow V_\rho$ lifts to a surjection $(\pi^*M)^{\vee\vee} \twoheadrightarrow V_\rho \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}$ of $\mathcal{O}_{Y,y}[G]$-modules. This must be an isomorphism and the claim follows by Proposition 4.4.

The last statement follows from Theorem 3.1. $\square$

Definition 5.6. A reflexive module M in $\mathsf{Ref}^{\mathsf{i}}_{X,x}$ will be called an *invariant module*.

Remark 5.7. It follows from the theorem that there is a one-to-one correspondence between isomorphism classes of invariant modules and conjugacy classes of profinite representations of the local fundamental group. In particular; for a quotient singularity (as first proved by Herzog, see [13]) there is a one-to-one correspondence between representations of the local fundamental group and reflexive modules, since a reflexive module on $(\mathbb{C}^2, 0)$ is free.

Corollary 5.8. *Any invariant module admits an integrable connection and a left $\mathcal{D}_{X,x}$-module structure. In particular; if M is a rank one (resp. any) reflexive module on a rational (resp. quotient) surface singularity, then M admits an integrable connection and a left $\mathcal{D}_{X,x}$-module structure.*

Proof. It is known that the rank one modules on a rational surface singularity have finite order with respect to the operation $[\otimes]$, see [17], and that $\pi : Y := \operatorname{Specan}((\oplus_n M^{\otimes n})^\vee) \to X$ is a finite Galois covering such that $\pi^*M \cong \mathcal{O}_Y$. $\square$

Not all reflexive modules are invariant modules. In fact, recall the definition of the fundamental module $E = E_{X,x}$ on (X,x). It is uniquely given by a non-trivial extension

$$0 \to \omega_{X,x} \to E \to \mathfrak{m}_{X,x} \to 0.$$

We have

Theorem 5.9. *Let (X,x) be a normal surface singularity. Then the fundamental module $E_{X,x}$ is an invariant module if and only if (X,x) is a quotient singularity.*

Proof. If (X,x) is a quotient singularity, all modules are invariant modules, so for the contrary assume that $E_{X,x}$ is an invariant module.

We first prove that for a Galois covering $\pi : (Y, y) \to (X, x)$ we have that $\left(\pi^* E_{X,x}\right)^{\vee\vee} \cong E_{Y,y}$. Since $V = Y \setminus \{y\} \to X \setminus \{x\} = U$ is étale, we have $\Omega^1_{V/U} = 0$ and thus the natural map $\pi^*\Omega^1_U \to \Omega^1_V$ is surjective and hence an isomorphism. From this we get that $\left(\pi^*\omega_{X,x}\right)_{|V} = \left(\omega_{Y,y}\right)_{|V}$. Thus we have that $\left(\pi^*\omega_{X,x}\right)^{\vee\vee} \cong \omega_{Y,y}$. Pulling back the fundamental sequence $0 \to \omega_{X,x} \to E_{X,x} \to \mathfrak{m}_{X,x} \to 0$ and taking dual twice, we get an exact sequence $0 \to \left(\pi^*\omega_{X,x}\right)^{\vee\vee} \to \left(\pi^* E_{X,x}\right)^{\vee\vee} \to \mathcal{O}_{Y,y} \to N \to 0$, with N supported on $\{y\} \subset Y$. Thus the kernel of $\mathcal{O}_{Y,y} \to N$ is an $\mathfrak{m}_{Y,y}$-primary ideal $\mathfrak{q}$. By properties of the fundamental module, see Lemma 11.10 in [22], we get a diagram:

$$\begin{array}{ccccccccc} 0 & \to & \left(\pi^*\omega_{X,x}\right)^{\vee\vee} & \to & \left(\pi^* E_{X,x}\right)^{\vee\vee} & \to & \mathfrak{q} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \omega_{Y,y} & \to & E_{Y,y} & \to & \mathfrak{m}_{Y,y} & \to & 0 \end{array}$$

We thus get that $\left(\pi^* E_{X,x}\right)^{\vee\vee} \xrightarrow{\varphi} E_{Y,y}$ is injective and the cokernel is supported on $\{y\} \subset Y$. But since both $\left(\pi^* E_{X,x}\right)^{\vee\vee}$ and $E_{Y,y}$ are reflexive, we see for instance by considering sections over $Y \setminus \{y\}$, that φ must be an isomorphism.

From Theorem 2.1 in [23] it follows that the fundamental module $E_{Y,y}$ is free if and only if (Y, y) is smooth, and hence (X, x) is a quotient singularity. $\square$

Remark 5.10. Let $\mu(M)$ denote the minimal number of generators of a finitely generated module M. We have that $\mu(E) \geq e$, where $e = e(X, x)$ is the embedding dimension. If we consider $B(E) = \varinjlim_{\pi} (\pi^* E)^{\vee\vee} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$ as in Definition 5.3, we see from the proof of the theorem that $B(E) = \varinjlim_{\pi} E_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$. There are infinite sequences of Galois maps between simple elliptic surface singularities, with increasing embedding dimension. Thus $B(E)$ is not a finite dimensional vector space.

In his thesis Kahn proved an interesting result (see [14, Cor. 6.16], chapter 6). By our Theorem 3.1 his result implies the following:

Theorem 5.11. *Let (X, x) be a simple elliptic surface singularity. For any reflexive $\mathcal{O}_{X,x}$-module M there is an integrable connection $\nabla : \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x}) \to \mathrm{End}_{\mathbb{C}}(M)$.*

Kahn explicitly exhibited representations of the fundamental group corresponding to each reflexive module. If (X, x) is simple elliptic of type $\mathrm{El}(b)$, the set of isomorphism classes of reflexive modules is in one-to-one correspondence with the set

$$\{(r, d, \lambda) \in \mathbb{N} \times \mathbb{N} \times \mathrm{Pic}^0(E) | r \leq d < (b + 1)r\}$$

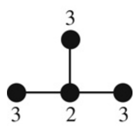
where E is the exceptional curve in the minimal good resolution, see Bemerkung 5.17.(1) in [14]. For each admissible pair (r, d) there exists a distinguished reflexive module $F_{r,d}$ of rank r such that:

(1) For every reflexive module M there is an isomorphism $M \cong F_{r,d}[\otimes]L$ for a rank one reflexive module L.
(2) $F_{r,d}$ is an invariant module if and only if $(r, d) = 1$.
(3) If $h = (r, d)$, there is an h-dimensional moduli space of isomorphism classes of integrable connections on $F_{r,d}$, see Theorem 6.30 in [14].

Remark 5.12. Note in particular that there is a one dimensional space of integrable connections on $F_{1,1} = \mathcal{O}_{X,x}$. However; only one of these corresponds to a profinite representation.

Example 5.13. Let $Y = \mathrm{Specan}(S)$ where $S = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$. Let $G = \langle\omega\rangle$ where ω is a primitive third root of unity. An action of G on S given by $(x, y, z) \mapsto (\omega x, \omega^2 y, \omega z)$ and we let X be the quotient of Y under this action.

The quotient X is an elliptic quotient and Y is its canonical cover. The graph of X is



There are $27r$ distinguished indecomposable reflexive modules on X of rank r.

(1) If $r \not\equiv 0 \bmod 3$ these are the only indecomposable reflexive modules on X of rank r.
(2) If $r \equiv 0 \bmod 3$ there are in addition $3r$ one-parameter families of non-isomorphic, indecomposable reflexive modules on X of rank r.

Using [17] on can show that the local fundamental group $\pi_1^{\text{loc}}(X, x)$ has two generators a and b with the following relations: $a^3 = b^3 = (a^2b^2)^3$. A calculation shows that there are 27 characters and 54 indecomposable rank two representations. Among the indecomposable rank two representations, 27 are simple and invariant modules and 27 are indecomposable but neither simple nor invariant modules, see [12].

6. Algebraisation

If R is a $\mathbb{C}$-algebra of finite type with a distinguished maximal ideal $\mathfrak{m}$, let R^{an} and R^{h} be the corresponding analytic and Henselian localisations of R in $\mathfrak{m}$. For any ring R we denote by Ref_R the category of reflexive R-modules M, and by Ref_R^{∇} the category of pairs (M, ∇) where M is a reflexive R-module and $\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M)$ is an R-linear map that satisfies the Leibniz rule and is a $\mathbb{C}$-Lie-algebra homomorphism.

Recall the functor $F : \mathsf{Rep}_{\pi_1^{\text{loc}}(X,x)} \to \mathsf{Ref}_{X,x}^{\nabla}$ from Section 3, defined in the analytic category. Restricting to profinite representations and invariant modules, we may also define a functor in the algebraic setting: Let $\mathsf{Ref}_R^{\mathsf{i},\nabla}$ be the full subcategory of the category Ref_R^{∇} consisting of reflexive modules given as $\left(\pi_* \left(V_\rho \otimes \mathcal{O}_Y\right)\right)^{G(Y/X)}$ for a Galois covering $\pi : Y = \operatorname{Spec} S \to \operatorname{Spec} R$ and a representation ρ of $G(Y/X)$, with the connection defined by restricting the canonical connection on $\pi_* V_\rho \otimes S^{\text{h}}$ to invariants. This connection is uniquely defined and will be called *the invariant connection.*

We define a functor

$$C : \mathsf{Rep}_{\widehat{\pi}_1^{\text{loc}}(X,x)} \to \mathsf{Ref}_{R^{\text{h}}}^{\mathsf{i},\nabla}$$

by letting $C(\rho) := (((\pi_\rho)_*(V_\rho \otimes S^{\text{h}}))^{\operatorname{im}\rho}, \nabla)$ and where ∇ is the invariant connection, and where $\pi_\rho : \operatorname{Spec} S^{\text{h}} \to \operatorname{Spec} R^{\text{h}}$ is the Galois covering corresponding to the subgroup $\ker\rho$ of $\pi_1^{\text{loc}}(X, x)$.

Theorem 6.1. *Let (X, x) be a normal surface singularity. Then the following holds:*

(1) *There exists a $\mathbb{C}$-algebra R of finite type, a maximal ideal $\mathfrak{m}$ in R and a map $R \to \mathcal{O}_{X,x}$ such that $R^{\text{an}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$.*
(2) *For every such algebraisation, the functor $M \mapsto M^{\text{an}} := \mathcal{O}_{X,x} \otimes_{R^{\text{h}}} M$ induces a bijection of isomorphism classes of objects in $\mathsf{Ref}_{R^{\text{h}}}$ and $\mathsf{Ref}_{X,x}$.*
(3) *The functor C is an equivalence of categories $\mathsf{Rep}_{\widehat{\pi}_1^{\text{loc}}(X,x)} \simeq \mathsf{Ref}_{R^h}^{\mathsf{i},\nabla}$ and hence $\mathsf{Ref}_{R^h}^{\mathsf{i},\nabla} \simeq \mathsf{Ref}_{X,x}^{\mathsf{i},\nabla}$.*

Proof. Because (X, x) is an isolated singularity (1) follows from Artin's Approximation Theorem ([2, Th. 3.8] and [2, Cor. 2.5]), and since reflexive modules on (X, x) are locally free on the complement of the singularity, the existence part of (2) is proved in [8, Th. 3]. Injectivity on isomorphism classes follows by [16, 7.11].

To prove (3) we note that by definition of the category $\mathsf{Ref}_{R^{\text{h}}}^{\mathsf{i},\nabla}$, we only need to prove that C gives an isomorphism on the Hom's. However; by considering partial internal Hom's we have that $\mathrm{Hom}((N_1, \nabla_1), (N_2, \nabla_2)) = \ker\nabla_Y$ where ∇_Y is the canonical connection on $\mathrm{Hom}_{S^{\text{h}}}(N_1, N_2)$, see Section 3.3, and $N_i = S^{\text{h}} \otimes V_{\rho_i}$. Here $Y = \operatorname{Spec} S^{\text{h}} \to \operatorname{Spec} R^{\text{h}}$ is a Galois covering dominating π_{ρ_1} and π_{ρ_2}. From Proposition 4.4, we get $\mathrm{Hom}((M_1, \nabla_1), (M_2, \nabla_2)) = \ker\nabla_X = (\ker\nabla_Y)^G$ where ∇_X is the invariant connection on $\mathrm{Hom}_{R^{\text{h}}}(M_1, M_2)$, $M_i = N_i^G$. But $(\ker\nabla_Y)^G = \mathrm{Hom}(V_{\rho_1}, V_{\rho_2})^G$. $\square$

Remark 6.2. It is not true that we have an equivalence of categories $\mathsf{Rep}_{\pi_1^{\text{loc}}(X,x)} \simeq \mathsf{Ref}_{R^{\text{h}}}^{\nabla}$: Let $R = \mathbb{C}[x, y]$, and consider the family of integrable connections on R^{h} given by $\partial/\partial x \mapsto \partial/\partial x + a$ and $\partial/\partial y \mapsto \partial/\partial y + a$, $a \in \mathbb{C}$. These connections have no nontrivial horizontal sections for $a \neq 0$, but have horizontal sections for $a = 0$. Thus there are non-isomorphic connections on R^{h}.

Acknowledgement

The authors are grateful for financial support from RCN's Strategic University Program in Pure Mathematics at the Department of Mathematics, University of Oslo (No. 154077/420).

References

[1] Revêtements étales et groupe fondamental (SGA 1), Documents Mathématiques (Paris) (Mathematical Documents (Paris)), 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. (Algebraic Geometry Seminar of Bois Marie 1960–61), Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original (Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)). MR MR2017446 (2004g:14017).

[2] M. Artin, Algebraic approximation of structures over complete local rings, Inst. Hautes Études Sci. Publ. Math. (36) (1969) 23–58. MR MR0268188 (42 #3087).

[3] Michael Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966) 129–136. MR MR0199191 (33 #7340).

[4] Maurice Auslander, Rational singularities and almost split sequences, Trans. Amer. Math. Soc. 293 (2) (1986) 511–531. MR MR816307 (87e:16073).

[5] Kurt Behnke, On Auslander modules of normal surface singularities, Manuscripta Math. 66 (2) (1989) 205–223. MR MR1027308 (90k:14031).

[6] I.N. Bernšteĭn, I.M. Gelfand, S.I. Gelfand, Differential operators on a cubic cone, Uspehi Mat. Nauk 27 1 (163) (1972) 185–190. MR MR0385159 (52 #6024).

[7] Jan-Erik Björk, Analytic $\mathcal{D}$-modules and applications, in: Mathematics and its Applications, vol. 247, Kluwer Academic Publishers Group, Dordrecht, 1993. MR MR1232191 (95f:32014).

[8] Renée Elkik, Solutions d'équations à coefficients dans un anneau hensélien, Ann. Sci. École Norm. Sup. (4) 6 (1973) (1974) 553–603. MR MR0345966 (49 #10692).

[9] Hélène Esnault, Reflexive modules on quotient surface singularities, J. Reine Angew. Math. 362 (1985) 63–71. MR MR809966 (87e:14033).

[10] H. Grauert, Th. Peternell, R. Remmert (Eds.), Several Complex Variables. VII, in: Encyclopaedia of Mathematical Sciences, vol. 74, Springer-Verlag, Berlin, 1994, Sheaf-Theoretical methods in complex analysis, in: A reprint of Current Problems in Mathematics. Fundamental Directions, vol. 74, Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow. MR MR1326617 (96k:32001) (in Russian).

[11] A. Grothendick, J. Dieudonné, Élements de géométrie algébrique, Inst. Hautes Études Sci. Publ. Math. 32 (1967).

[12] Trond Stølen Gustavsen, Runar Ile, Reflexive modules on log-canonical surface singularities. http://www.math.uio.no/˜stolen, 2006, Preprint.

[13] Jürgen Herzog, Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen–Macaulay–Moduln, Math. Ann. 233 (1) (1978) 21–34. MR MR0463155 (57 #3114).

[14] Constantin P.M. Kahn, Reflexive Moduln auf einfach-elliptischen Flächensingularitäten, Bonner Mathematische Schriften (Bonn Mathematical Publications), 188, Universität Bonn Mathematisches Institut, Bonn, 1988, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1988 MR MR930666 (90h:14045).

[15] Nicholas M. Katz, An overview of Deligne's work on Hilbert's twenty-first problem, in: Mathematical Developments Arising from Hilbert Problems (Proc. Sympos. Pure Math., vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), Amer. Math. Soc., Providence, RI, 1976, pp. 537–557. MR MR0432640 (55 #5627).

[16] Hideyuki Matsumura, Commutative Ring Theory, second ed., in: Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989 (Translated from the Japanese by M. Reid). MR MR1011461 (90i:13001).

[17] David Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Inst. Hautes Études Sci. Publ. Math. (9) (1961) 5–22. MR 27 #3643.

[18] Constantin Năstăsescu, F. Van Oystaeyen, Graded and filtered rings and modules, in: Lecture Notes in Mathematics, vol. 758, Springer, Berlin, 1979. MR R551625 (80k:16002).

[19] Walter D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc. 268 (2) (1981) 299–344. MR MR632532 (84a:32015).

[20] Jean-Pierre Vigué, Opérateurs différentiels sur des cônes normaux de dimension 2, Bull. Soc. Math. France 103 (2) (1975) 113–128. MR MR0447616 (56 #5926).

[21] Philip Wagreich, Singularities of complex surfaces with solvable local fundamental group, Topology 11 (1971) 51–72. MR MR0285536 (44 #2754).

[22] Yuji Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, in: London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990. MR MR1079937 (92b:13016).

[23] Yuji Yoshino, Takuji Kawamoto, The fundamental module of a normal local domain of dimension 2, Trans. Amer. Math. Soc. 309 (1) (1988) 425–431. MR MR957079 (89h:13033).